Canberra Times (1965-1968) and After

by **Eugene Seneta**

"Say, is there Beauty yet to find?

And Certainty? And Quiet kind?

...oh! Yet

Stands the Church clock at ten to three?

And is there honey still for tea?"

Rupert Brooke (Café des Westens, Berlin, May 1912)

From "The Old Vicarage, Grantchester."

1 1964. Quasi-stationary Distributions

Suppose the transition matrix of a Markov chain $X = (X_n : n \geq 0)$ on $S = \{0, 1, 2, \cdots\}$ is

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & T \end{array} \\ \begin{array}{c} 0 \\ T \end{array} & \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{p_0} & Q \end{pmatrix} \end{array}, \text{ where } Q\mathbf{1} \neq \mathbf{1}.$$

Then

$$\lim_{n\to\infty} \mathbb{P}(X_n = j | X_n > 0) \longrightarrow v_j, \;\; j \in T = \{1, 2, 3, \cdots\} \tag{1}$$

where

$$\mathbf{v}^T Q = \rho \mathbf{v}^T, \text{where } \mathbf{v}^T\mathbf{1} = \sum_j v_j = 1, \tag{2}$$

irrespective of the initial distribution $\pi^T = \{\pi_j,\}$, $j \in \{1, 2, 3, \cdots\}$ over T. □

Darroch, J. N. and Seneta,E. (1965) On quasi-stationary distributions in absorbing discrete-time finite Markov chains. *Journal of Applied Probability*, **2**, 88-100.

2 1965

Let $\{Z_n\}, n \geq 0, Z_0 = 1$, be the simple branching (Bienaymé - Galton - Watson) process, with offspring pgf $F(s)$, and $0 < m = F'(1-) < 1$. Then

$$\lim_{n\to\infty} \mathbb{P}(Z_n = j | Z_n > 0) \longrightarrow v_j, \; j \in T = \{1, 2, 3, \cdots\} \tag{3}$$

where $A(s) = \sum_{j=1}^{\infty} v_j s^j, \; 0 \leq s < 1$, satisfies $A(1-) = 1$, and

$$1 - A(F(s)) = m(1 - A(s)), \; A(0) = 0, \tag{4}$$

and $A(s)$ is the unique pgf solution of this equation.

Due to Yaglom (1947) under additional moment conditions on the offspring distribution.

Heathcote, C.R., Seneta, E. and Vere-Jones, D. (1967) A refinement of two theorems in the theory of branching processes. *Teoriia Veroiatnostei i ee Primeneniia*, **12**, 311-316.

Seneta,E. and Vere-Jones, D. (1966) On quasi-stationary distributions in discrete time Markov chains with a denumerable infinity of states.*Journal of Applied Probability*, **3**, 403-434.

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$$\lim_{n\to\infty} \mathbb{P}(Z_n = j | Z_n > 0) \longrightarrow v_j, \; j \in T = \{1, 2, 3, \cdots\} \tag{5}$$

where $A(s) = \sum_{j=1}^{\infty} v_j s^j, \; 0 \leq s < 1$, satisfies $A(1-) = 1$, and

$$1 - A(F(s)) = m(1 - A(s)), \; A(0) = 0, \tag{6}$$

and $A(s)$ is the unique pgf solution of this equation.

Figure 1: Boris V.Gnedenko, Mary and David Vere-Jones, Ludmilla Seneta. (Oct., 1965)

2. If Z_0 has an initial distribution $\pi^T = \{\pi_j,\}$, $j \in \{1, 2, 3, \cdots\}$,with pgf $\pi(s) = \sum_{j=1}^{\infty} \pi_j s^j$, other proper conditioned limit distributions may arise: in particular ones whose pgf $B(s)$ satisfies:

$$1 - B(F(s)) = m^{\alpha}(1 - B(s)),\ \ B(0) = 0 \tag{7}$$

for some $\alpha, 0 < \alpha < 1$, for example $B(s) = 1 - (1 - A(s))^{\alpha}$.

Rubin and Vere-Jones (1968): A **necessary and sufficient condition** on $\pi(s)$ to obtain the conditioned limit distribution with pgf

$$B(s) = 1 - (1 - A(s))^{\alpha},\ \ 0 < \alpha \le 1,$$

is

$$1 - \pi(1 - u) = u^{\alpha} L(u)$$

where $L(u)$ is slowly varying at 0. $\square$

Fred M. Hoppe (1980):

1. The pgf solution $B(s)$ to (7) is not unique when $0 < \alpha < 1$.
2. An initial distribution with pgf $\pi(s)$ gives arise to a limiting conditional distribution ("Yaglom limit") with pgf $B(s)$ (which then satisfies (7) for *some* $0 < \alpha \le 1$) if and only if

$$1 - \pi(1 - u) = (1 - B(1 - u))L(u) \tag{8}$$

where $L(u)$ is slowly varying at 0. $\square$

Figure 2: Aleksandr Ya.Khinchin (1894-1959)

3 Autocorrelation and Real Characteristic Functions

The following lemma shows the equivalence between autocorrelation functions and real characteristic functions. It depends on Khinchin's Theorem for the proof in one direction. In the other direction it is suggested by Section 6.11 of Bartlett (1960).

Lemma 1. *A continuous function $\rho(s)$, $s \in \mathbb{R}$, with $\rho(0) = 1$, is the autocorrelation function of a (real valued) second-order stationary process in continuous time if and only if $\rho(s)$ is a real-valued characteristic function. The cumulative distribution function corresponding to this characteristic function then describes the normed spectral measure corresponding to the autocorrelation function $\rho(s)$.*

Proof. If $\rho(s)$, $s \in \mathbb{R}$ is a real-valued characteristic function, then

$$\rho(s) = \int_{-\infty}^{\infty} \cos(sx)\, dF(x) \tag{9}$$

where $F(x) = \mathbb{P}(X \leq x)$ is the cumulative distribution function of a random variable X with distribution on $\mathbb{R}$ symmetric about 0.

Now consider the process $\{X(t)\}$, $t \in \mathbb{R}$ given by

$$X(t) = \sqrt{2}\cos(V + Wt)$$

where $V \sim Unif(0, 2\pi)$ and independently distributed of W whose cumulative distribution function is $F(w)$, $w \in \mathbb{R}$. Then

$$\mathbb{E}(X(t)) = 0,\ \mathbb{C}\mathrm{ov}(X(t), X(s)) = \mathbb{E}(\cos((t-s)W))$$

so the process $\{X(t)\}$ is second-order stationary, and $\mathbb{V}\mathrm{ar}(X(t)) = 1$. Hence, the autocorrelation function of $X(t)$ is

$$\rho(\tau) = \int_{-\infty}^{\infty} \cos(\tau w)\, dF(w)$$

which is consistent with Equation (9). The cumulative distribution function $F(w)$ describes the normed spectral measure of the process on account of Equation (9).

Conversely, if $\rho(s)$, $s \in \mathbb{R}$ is the continuous autocorrelation function of a second-order stationary process, then by the **Khinchin's** Theorem of 1934 for real autocorrelation functions (in a complex value setting called the Wiener-Khinchin Theorem)

$$\rho(\tau) = \int_{-\infty}^{\infty} \cos(\tau w)\, dF(w)$$

for some cumulative distribution function $F(w)$ of a symmetric distribution and so $\rho(\tau)$ is a real-valued characteristic function. $\square$

Corollary 1. *Any cumulative distribution function $F(w)$ corresponding to a probability distribution which is symmetric about 0 on $\mathbb{R}$ describes a normed spectral measure corresponding to a continuous autocorrelation function. The relation is given by Equation (9).*

4 Stochastic Processes with Convex ACF

Definition 1. A sequence $a_0, a_1, \ldots$ is said to be convex if $a_{\nu+1} - 2a_\nu + a_{\nu-1} \geq 0$, $\nu \geq 1$.

Theorem 1. *If $a_\nu \to 0$ and the sequence $a_0, a_1, \ldots$ is convex, then the series*

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} a_\nu \cos(\nu x) \tag{10}$$

converges, except possibly at $x = 0$, to a non-negative, integrable and continuous function $f(x)$, and is the Fourier series of f.

Due to Young (1913) and Kolmogoroff (1923) (Zygmund, 1955).

Assumption. $\rho(s)$ is a function symmetric on $\{0, \pm 1, \pm 2, \ldots\}$ satisfying on $\{0, 1, 2, \ldots\}$: $\rho(0) = 1$, $\rho(s) \geq 0$, $\rho(s) - \rho(s+1) \geq 0$ and $\{\rho(s)\}$ is convex.

Theorem 2. *If $\{\rho(k)\}$, $k = 0, 1, 2, \ldots$ satisfies the Assumption, and $\rho(s) \downarrow \delta \geq 0$, then, putting*

$$\rho^*(s) = \frac{\rho(s) - \delta}{1 - \delta},$$

it follows that that the series

$$\frac{1}{2}\rho^*(0) + \sum_{k=1}^{\infty} \rho^*(k) \cos(kx) \tag{11}$$

converges, except possibly at $x = 0$, to a non-negative integrable function $f^*(x)$ which is continuous everywhere except possibly at $x = 0$, and is the Fourier series of f^*.

Furthermore, $\rho(s)$ is an autocorrelation function of a second-order stationary process on $s \in \{0, \pm 1, \pm 2, \ldots\}$. This process has normed spectral measure described by an atom of probability δ at the origin, and density $(1 - \delta)\frac{1}{\pi} f^*(x)$, $-\pi < x \leq \pi, x \neq 0$.

It is imbedded in a second-order stationary process on $\mathbb{R}$ whose acf $\rho(s)$ is the characteristic function of this distribution.

5 Long Range Dependence (LRD)

In general, for a given function $\rho(s)$ satisfying our Assumption, we can obviously construct Gaussian processes; having the specified autocorrelation structure; and simple processes of the kind figuring in Lemma 1.

One of the common criteria for describing LRD (long range dependence), is

$$\sum_{j=0}^{\infty} \rho^*(j) = \infty \tag{12}$$

sometimes we need to construct stochastic processes which are not only LRD, but have additional features such as asymptotic self-similarity.

Under our **Assumption**: when $0 < \alpha < 1$, $A > 0$:

$$f^*(x) \sim \frac{A\pi}{2\Gamma(\alpha)\cos(\alpha\pi/2)} x^{\alpha-1} L(1/x) \text{ as } x \to 0+ \tag{13}$$

$$\Longleftrightarrow$$

$$\rho^*(n) \sim An^{-\alpha}L(n), \text{ as } n \to \infty \tag{14}$$

where $L(x)$ is slowly varying as $x \to \infty$.

Thus if $0 < \alpha < 1$, and either of the equivalent conditions (13), (14) holds, we have (12).

In particular, we have a range of real characteristic functions to choose from, to satisfy (14), whose behaviour is reflected by that of (13) at the origin.

Theorem 3. *There exists a strictly stationary non-Gaussian process* $\{X_t\}$, $t \in \mathbb{N}$ *with given infinitely divisible marginal distribution with finite variance such that* $\mathbb{C}or(X_t, X_{t+s}) = \rho(s)$ *for* $s \in \mathbb{N}$, *where* $\rho(s)$ *is any function satisfying the Assumption.*

Due to Richard Finlay (Finlay and Seneta, 2007), and simplified.

Theorem 2 also helps us understand better the following analytical result (Theorem 2 of Dietrich and Newsam ,1997).

Theorem 4. *If the entries of the vector* $r = (r_0, r_1, \cdots, r_m)$ *form a sequence that is convex, decreasing and nonnegative , then* r *has a nonnegative definite minimal embedding.*

This is actually a consequence of our Theorem 2, if we take $\rho(s) = \frac{r_s}{r_0}, s = 0, 1, \cdots m; = \frac{r_m}{r_0}, s > m$, assuming $r_m > 0$, in which case $\delta = \frac{r_m}{r_0}$.

The crucial "nonnegativity" aspect, in our terms, is the fact that

$$f^*(\frac{2\pi j}{2m}) \geq 0.$$

Gneiting (2000) says

> A particularly attractive technique for the simulation of Gaussian processes has been proposed in the appendix of Davies and Harte (1987) ... the method depends on a circulant embedding of the covariance matrix ... it works only

if the circulant embedding results in nonnegative definite covariance matrix... Davies and Harte devised the circulant embedding technique for the simulation of fractional Gaussian noise with correlation function

$$\rho(v) = \frac{1}{2}(|v+1|^{2H} - 2|v|^{2H} + |v-1|^{2H}), \ (0 < H < 1) \quad (15)$$

where H is the Hurst coefficient.

Davies, R.B. and Harte, D.S. (1987) Tests for Hurst effect. *Biometrika,* **74,** 95-101.

Gneiting(2000) shows that if $H \in [1/2, 1)$, (15) satisfies our Assumption, and so Theorem 4 applies, which confirms that for these values of H fractional Brownian motion always allows for exact simulation with the circulant embedding technique.

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6 Appendix. On David's 65th Birthday Celebration

David Vere-Jones: Some Reminiscences

Eugene Seneta

19 April, 2001

I've always been pleased to speak of David as my de-facto supervisor for Ph.D.

Our contact began early in 1965 when he took up an appointment in Pat Moran's Department at the Institute of Advanced Studies at the Australian National University (A.N.U.) in Canberra. Pat was rebuilding the Department, which had become a focus for students wanting to do a Ph.D. in Statistics (in the broad sense). Other staff to arrive that year were Nick Day, Roger Miles and Michael Hasofer. Don McNeil (now Professor at Macquarie University) and I were the new Ph.D. students, together with C.K. Cheong (now Deputy Chairman and Executive Officer of Singapore Airlines).

There were two Departments of Statistics at A.N.U. then. The other was the undergraduate teaching department under Ted Hannan in the School of General Studies on the other side of the campus, where I was located as Temporary Senior Tutor.

Cheong and I had gone through the same courses during our undergraduate years at Adelaide University. He became David's first official Ph.D. student. I'd brought with me from Adelaide an interest in finite non-negative matrices and an M.Sc. (under John Darroch) on quasi-stationary distributions in finite Markov chains.

David, through his development of the R-theory of countable non-negative matrices (although his major papers had yet to be written) had the knowledge, insight, and most important, willingness to guide and help me in the application of this theory to settings of quasi-stationary distributions and branching processes in denumerably infinite Markov chains.

We all know that David loves to travel. Before he came to A.N.U. he had already travelled much. After his Oxford D.Phil., he had spent some time in India and in Russia. In Moscow he had learned Russian and was greatly influenced, in probability and in mathematical education, by B.V. Gnedenko, who had done a great deal after World War 2 to build up probability theory in Ukraine. Since my own background is Ukrainian and my wife is Russian-born, this contributed a great deal to my empathy with David.

In mathematics I saw him as directly in the English tradition of analytical theory of Markov processes, in the line of David Kendall, Harry Reuter and John Kingman. It was in this tradition that I remember some advice from him on writing a paper:

> "You must have something to say. After that, style is everything."

And while I'm speaking of David as a teacher: at another time I said that I couldn't understand measure-valued branching processes, to which he responded, completely accurately:

> "You've never tried."

Both facts remain true to this day.

In Canberra, David initially lived at University House, where Charlie Pearce, another New Zealander, was notable for his automotive activities. Later that first year at A.N.U., David married Mary and they moved to Boronia Crescent, O'Connor. I remember Sunday mornings when we enjoyed omelettes, one of David's culinary specialties, interspersed with mathematical gossip. I have photographs taken in their garden from the visit of Gnedenko to A.N.U. in 1965, among the many happy social occasions we enjoyed there. Gnedenko's influence is an ongoing link between us still, for me because of Gnedenko's writings on the history of probability.

My wife and I were sad to see the Vere-Joneses leave Canberra in 1969, but life moves on, and we, too, left for similar reasons in 1979.

There were intermittent meetings. One particularly interesting one which brought David back to India was the 13th Conference on Stochastic Processes and Their Applications, at Banaras Hindu University at Varanasi (Benares) in 1983. Joe Gani was present when I slipped and fell in the mud near a fountain when we were inspecting a Hindu temple. Our local colleagues with great concern wanted to rush me off to hospital, but I didn't think that was necessary (or advisable). David and I had met in New Delhi before the conference and shared a meal of emaciated tandoori chicken. He said that irrespective of precautions, he always got sick after two weeks in India. I remember Geof Watson thinking me foolhardy to have icecream after a meal at the hotel in Varanasi where David and I shared a room. One morning typically David organized for us to see the sunrise over the Ganges at Varanasi from a boat; this produced the most beautiful photographs I have ever taken. Very interesting too was the conference itself. During one session he and I were sitting in the front row when a very nervous young man speaking on branching processes managed to finish his presentation in about a half of the allocated time of 25 or so minutes.

David remarked to me quietly:

> "He should reduce his firing rate."

Another speaker, an eminent Central-European academic, stood to one side of the projector with arm outstretched pointing to the screen for at least 10 minutes without moving and delivered a monologue in a high-pitched monotone. David listened with his customary courtesy, but afterwards confided that it might have been considerably better presented (at least, words to that effect but in stronger vein!).

The theme of travel is thus an appropriate one on which to conclude. Augustus De Morgan, the English probabilist whose name occurs in De Morgan's Laws, when writing in about 1850 on the dates on which Easter is celebrated, says:

> "This table goes from 1850 to 1999: should the New Zealander not have arrived by that time, and should the churches of England and Rome then survive, the epact table may be continued from their liturgy books."

Well, Easter of 2001 has just come and gone, the churches of Rome and of England certainly survive, and *this* New Zealander has yet *far to go* in gracing the statistical and seismological world.

May he long continue to enrich lives as he has enriched ours.