Adv. Appl. Prob. **9**, 724–746 (1977)
Printed in Israel
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TESTING THE HYPOTHESIS THAT A POINT PROCESS IS POISSON

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Abstract

The testing of the hypothesis that a point process is Poisson against a one-dimensional alternative is considered. The locally optimal test statistic is expressed as an infinite series of uncorrelated terms. These terms are shown to be asymptotically equivalent to terms based on the various orders of cumulant spectra. The efficiency of tests based on partial sums of these terms is found.

ASYMPTOTIC EFFICIENCY; CUMULANT; CUMULANT SPECTRA; LOCAL EFFICIENCY; LOCAL OPTIMALITY; PERIODOGRAM; POINT PROCESS; POISSON PROCESS

1. Introduction

Numerous tests have been proposed for testing the hypothesis that a point process is Poisson against general not precisely specified alternatives; see, for example, Cox and Lewis (1966), §6.3. In contrast, in this paper we investigate tests designed for specific alternatives. We suppose that we have a probability model for the process depending on just two parameters, $\xi \geqq 0$ and $\mu > 0$, such that if $\xi = 0$ the process is Poisson with rate μ. The test that the process is Poisson then becomes a test of the hypothesis $\xi = 0$ against the alternative $\xi > 0$.

Our starting point is the locally optimal test obtained by differentiating the log-likelihood. However, for many point process models the likelihood function is too complex to be evaluated and it would be preferable to base the test on the cumulants of the process. In Section 2 we differentiate Kuznetsov and Stratonovich's (1956) formula for the likelihood and obtain the locally optimal test statistic as the sum of an infinite series of terms, each term being based on a different order of cumulant and, in effect, providing a different piece of information. Of particular interest is the efficiency of tests based on only a few of these terms.

In Section 3 we show that each term is asymptotically equivalent to a term

Received in revised form 30 December 1976.
Part of this work was carried out while the author was a guest of the Statistical Laboratory, University of California, Berkeley, and was partially supported by U.S.N.I.H. grant GM–10525.

based on the corresponding order of periodogram and cumulant spectrum. The results of this section, besides being of practical importance, are of interest in indicating how much of the information about ξ is contained in the second-order periodogram. In Section 4 we give several examples, and an appendix contains proofs of the main theorems.

The efficiencies of the tests are expressed as *local efficiencies* and *asymptotic local efficiencies.* These concepts, together with some theory on locally optimal tests, are discussed in the remainder of this section. Quite complicated expressions are involved in obtaining the results and it seems advisable to carry out the analysis with some rigour. By requiring the differentiations to exist in $L1$ or, in some cases, $L2$, the technical problems of interchanging orders of differentiation and integration have been avoided at the expense of slightly more complicated statements to the theorems. The reader who is not interested in these finer points should regard differentiable in $L1$ or $L2$ simply as differentiable.

Suppose that we observe a random variable X which is drawn from a population with density $p(X;\xi)$ with respect to some measure λ. We wish to test the hypothesis $\xi = 0$ against the alternative $\xi > 0$. A locally optimal test with significance level α is a test which maximizes the derivative of the power function at $\xi = 0$ amongst all level α tests. The following theorem follows from the Neyman–Pearson lemma.

Theorem 1.1. Suppose $p(X;\xi)$ is $L1$ differentiable at $\xi = 0$ in the sense that there exists $p'(X)$ such that

$$\lim_{\xi \to 0} \int |[p(X;\xi) - p(X;0)]/\xi - p'(X)| \, \lambda(dx) = 0. \tag{1.1}$$

Then the test with critical region

$$\{p'(X)/p(X;0) > c\} \tag{1.2}$$

where c has been adjusted to give a significance level α (with randomisation on the boundary if necessary) is locally optimal.

We are concerned with the efficiency of various other tests. Efficiencies in terms of derivatives of the power function would be very difficult to handle. However in the present paper the test statistics are approximately normally distributed and thus a useful measure of efficiency of a test can be made in terms of the derivative of the expectation of the test statistic.

Definition. Suppose

$$Z = p'(X)/p(X;0) \tag{1.3}$$

where $p'(X)$ is as in (1.1). Then the *local efficiency* of a test statistic T is defined as

$$\rho(T) = \left. \frac{\lim_{\Delta\to\infty} \left\{ \frac{\partial}{\partial\xi} E_\xi C_\Delta(T) \right\}^2 / \mathrm{Var}_0(T)}{\lim_{\Delta\to\infty} \left\{ \frac{\partial}{\partial\xi} E_\xi C_\Delta(Z) \right\}^2 / \mathrm{Var}_0(Z)} \right|_{\xi=0} \tag{1.4}$$

where E_ξ denotes expectation under $p(X;\xi)$, Var_0 denotes variance under $p(X;0)$ and

$$C_\Delta(t) = t \quad \text{if} \quad |t| < \Delta$$
$$0 \quad \text{if} \quad |t| \geqq \Delta.$$

The truncating at $\pm\Delta$ is to avoid some irrelevant regularity conditions. For the definition to be meaningful it is necessary that $\mathrm{Var}_0(Z) < \infty$. One can now prove the following theorem.

Theorem 1.2. Suppose Condition (1.1) holds and $\mathrm{Var}_0(Z) < \infty$. Then the local efficiency of a test statistic T with $\mathrm{Var}_0(T) < \infty$ is given by

$$\rho(T) = [\mathrm{Corr}_0(Z, T)]^2 \tag{1.5}$$

where Corr_0 denotes correlation under $p(X;0)$.

A corollary of this theorem shows that if S and T are two test statistics which are close together under the hypothesis then they have similar local efficiencies.

Corollary 1.3. Suppose S and T are test statistics with $\mathrm{Var}_0(S)$, $\mathrm{Var}_0(T) < \infty$, (1.1) is satisfied and $\mathrm{Var}_0(Z) < \infty$. Then

$$|[\rho(S)]^{\frac{1}{2}} - [\rho(T)]^{\frac{1}{2}}| \leqq 2[\mathrm{Var}_0(S - T)/\mathrm{Var}_0(T)]^{\frac{1}{2}}.$$

A further corollary provides the main result required by the next section.

Corollary 1.4. Suppose (1.1) is satisfied. Suppose further that Z as defined in (1.3), satisfies $\mathrm{Var}_0(Z) < \infty$ and can be expressed

$$Z = \sum_{q=1}^{\infty} Z_q \tag{1.6}$$

where the terms Z_q are mutually uncorrelated under $p(X;0)$, Z_q having variance τ_q^2. Suppose further that the statistic T with $\mathrm{Var}_0(T) < \infty$ can be expressed

$$T = \sum_{q=1}^{\infty} T_q$$

where the T_q are again uncorrelated and also T_q and Z_r are uncorrelated if $q \neq r$. Suppose $\mathrm{Corr}_0(T_q, Z_q) = \rho_q \geqq 0$ and $\mathrm{Var}_0(T_q) = \rho_q^2 \tau_q^2$. Then T has local efficiency

$$\rho(T) = \sum_1^\infty \rho_q^2 \tau_q^2 \Big/ \sum_1^\infty \tau_q^2. \tag{1.7}$$

In particular

$$\rho\left(\sum_{q \in Q} Z_q\right) = \sum_{q \in Q} \tau_q^2 \Big/ \sum_1^\infty \tau_q^2. \tag{1.8}$$

Thus each term Z_q, in effect, carries a separate piece of information, the 'amount of information' being proportional to τ_q^2. Similarly each T_q carries an amount of information proportional to $\rho_q^2 \tau_q^2$. Note that the variance assigned to T_q in Corollary 1.4 ensures that the T_q are scaled in the optimum way.

We also consider the properties of our tests as the interval of observation, t, tends to infinity. It would be possible to define the *asymptotic local efficiency* of a test statistic as the limit, as t tends to infinity, of the local efficiency. However, it would be preferable to base it on expectations and variances of the limiting distributions rather than the limits of expectations and variances. In this paper no problem arises with the statistic Z, but for the other test statistics considered it is appropriate to allow for the possibility that the limit of the variance is not equal to the variance of the limiting distribution. Thus our definition is based on local efficiency of truncated versions of the test statistic.

Definition. Suppose T_t, a test statistic based on an interval of observation of length t, has been normalised so that as t tends to infinity, it has a proper non-degenerate limiting distribution under $\xi = 0$. Suppose that, with ρ and C_Δ as previously defined,

$$\lim_{\Delta \to \infty} \lim_{t \to \infty} \inf \rho[C_\Delta(T_t)] = \lim_{\Delta \to \infty} \lim_{t \to \infty} \sup \rho[C_\Delta(T_t)].$$

Then the *asymptotic local efficiency* will be defined as the common value of these expressions.

If T_t has the variance of its limiting distribution equal to the limit of its variance, then the asymptotic local efficiency is the limit of the local efficiency. However, unlike the limit of local efficiency, asymptotic local efficiency has the important property which follows from Corollary 1.3: if two test statistics have proper non-degenerate limiting distributions and are asymptotically equal when $\xi = 0$, then they have the same asymptotic local efficiency.

It is also necessary to consider tests in the presence of a nuisance parameter. Several approaches are possible; the simplest follows along the lines of Neyman (1959). Suppose that the density of the random variable is now denoted by $p(X; \xi, \mu)$ where μ is the nuisance parameter. Further suppose that there is a statistic U such that for each value of μ

$$\frac{\partial}{\partial \mu} \log p(X; 0, \mu) \tag{1.9}$$

is a linear function of U. We want the distribution of a test statistic T to be independent of μ and so we might require

$$\frac{\partial}{\partial \mu} E_{0,\mu}(T) = 0 \tag{1.10}$$

where $E_{\xi,\mu}$ denotes expectation under $p(X; \xi, \mu)$. It follows from Theorem 1.2 that if a differentiability condition similar to (1.1) holds then (1.10) will imply that T and U are uncorrelated. Thus to maximise the local efficiency of a test statistic, Z_1, subject to (1.10) we should choose.

$$Z_1 = Z - U\,\mathrm{Cov}_{0,\mu}(U, Z)/\mathrm{Var}_{0,\mu}(U). \tag{1.11}$$

Unfortunately Z_1 may still depend on μ. However we will see that in the present context, asymptotically, μ may be replaced by an estimate.

Analogues of Theorem 1.2 and Corollaries 1.3 and 1.4 will still hold when attention is restricted to random variables that satisfy (1.10) and local efficiency is with respect to Z_1 as defined in (1.11).

The asymptotic local efficiencies defined in this section are closely related to Pitman efficiencies, defined in terms of the observation time, t, and it is expected that they will be the same for the tests considered in this paper. See, for example, Kendall and Stuart (1961), §25.6.

2. Main results

We suppose that a point process is observed over the interval $(0, t)$ and that *points* occur at times $T_1, \cdots, T_n$ in this interval (n is a random variable). The outcome may also be expressed as an integer-valued measure, N, which assigns a measure 1 to each point at which a *point* occurs. Thus $N(a, b)$ equals the number of *points* occurring in the interval (a, b). Throughout this paper we suppose that the point process is orderly (only one *point* at a time) and finite (only a finite number of *points* in any finite interval of time).

Suppose that the moment density of the process can be defined:

$$m(t_1, \cdots, t_q) = \lim_{\tau \to 0} \tau^{-q} \Pr\{N(t_i, t_i + \tau) = 1;\ i = 1, \cdots, q\} \tag{2.1}$$

for each q distinct points $t_1, \cdots, t_q$. Since $m(t_1, \cdots, t_q)$ is invariant under permutations of the arguments and we are not concerned with its value when any of the arguments coincide it is convenient to regard m as a set function defined for finite sets and in particular we do not note any special dependence of m on the number of arguments q. The cumulant density may be defined $k(t_1, \cdots, t_q) =$ coefficient of $\Pi_1^q u_i$ in

$$\log\left[1 + \sum m(t_{\nu_1}, \cdots, t_{\nu_j}) \prod_{i=1}^{j} u_{\nu_i}\right] \tag{2.2}$$

where the sum is over all non-empty subsets $\{\nu_1, \cdots, \nu_j\}$ of $\{1, \cdots, q\}$. In practice k is usually calculated from the probability generating functional or from the formula

$$(2.3) \qquad k(t_1, \cdots, t_q) = \lim_{\tau \to 0} \tau^{-q} \frac{\partial}{\partial u_1} \cdots \frac{\partial}{\partial u_q} \log E\left[\prod_1^q u_i^{N(t_i, t_i+\tau)}\right]\Bigg|_{u_i=1}.$$

The value of k when two of its arguments coincide is immaterial and it may be defined arbitrarily, say by continuity.

Now suppose the model for the process depends on the two parameters ξ and μ such that when $\xi = 0$ the process is Poisson with rate μ. The dependence of m and k on these parameters will be indicated by including them in the argument lists, for example $k(t_1, \cdots, t_q; \xi, \mu)$. Thus

$$(2.4) \qquad k(t_1, \cdots, t_q; 0, \mu) = \begin{cases} 0 & \text{if} \quad q > 1 \\ \mu & \text{if} \quad q = 1. \end{cases}$$

For the small sample results we will require that k is differentiable at $\xi = 0$ in the sense that there exists $k'(t_1, \cdots, t_q; \mu)$ such that

$$(2.5) \qquad \lim_{\xi \to 0} \sum_{q=1}^{\infty} \frac{2^q}{q!} \int_{[0,t]^q} \Big| [k(t_1, \cdots, t_q; \xi, \mu) - k(t_1, \cdots, t_q; 0, \mu)]/\xi - k'(t_1, \cdots, t_q; \mu) \Big| \, dt_1 \cdots dt_q = 0$$

and further that

$$(2.6) \qquad \sum_{q=1}^{\infty} [\tau_q^{(t)}]^2 < \infty$$

where

$$(2.7) \qquad [\tau_q^{(t)}]^2 = \frac{1}{q!\,\mu^q} \int_{[0,t]^q} [k'(t_1, \cdots, t_q; \mu)]^2 dt_1 \cdots dt_q$$

and where we are denoting the multiple integral

$$\int_{t_1=0}^{t} \cdots \int_{t_q=0}^{t} \quad \text{by} \quad \int_{[0,t]^q}$$

For the asymptotic results we require the process to be stationary: for each $\mu > 0, \xi \geqq 0, q, t_1, \cdots, t_q, \tau$

$$(2.8) \qquad k(t_1 + \tau, \cdots, t_q + \tau; \xi, \mu) = k(t_1, \cdots, t_q; \xi, \mu).$$

We further require that there exists $k'(t_1, \cdots, t_q; \mu)$ satisfying the following conditions:

$$(2.9) \qquad k'(t_1 + \tau, \cdots, t_q + \tau; \mu) = k'(t_1, \cdots, t_q; \mu)$$

for each $t_1, \cdots, t_q, \tau$;

$$\lim_{\xi \to 0} \sum_{q=1}^{\infty} \frac{1}{q!\mu^q} \int_{\mathfrak{R}^{q-1}} \{[k(0, t_2, \cdots, t_q; \xi, \mu) - k(0, t_2, \cdots, t_q; 0, \mu)]/\xi - k'(0, t_2, \cdots, t_q; \mu)\}^2 dt_2 \cdots dt_q = 0; \tag{2.10}$$

$$\sum_{q=1}^{\infty} \tau_q^2 < \infty; \tag{2.11}$$

where

$$\tau_q^2 = \frac{1}{q!\mu^q} \int_{\mathfrak{R}^{q-1}} [k'(0, t_2, \cdots, t_q; \mu)]^2 dt_2 \cdots dt_q, \tag{2.12}$$

and for each q

$$\lim_{\mu_1 \to \mu} \int_{\mathfrak{R}^{q-1}} [k'(0, t_2, \cdots, t_q; \mu_1) - k'(0, t_2, \cdots, t_q; \mu)]^2 dt_2 \cdots dt_q = 0. \tag{2.13}$$

Conditions (2.10), (2.11) imply Conditions (2.5), (2.6).

We need the notation

$$\chi(t_1, \cdots, t_q) = \begin{cases} 1 & \text{if no arguments coincide,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.14}$$

For the moment we suppose that μ is known. The first theorem expresses the locally optimal test as an infinite series and forms the basis of this paper.

Theorem 2.1. Suppose the moment density function defined in (2.1) exists, $k(t_1, \cdots, t_q; \xi, \mu)$ is as defined in (2.2), (2.4) is satisfied and k is differentiable at $\xi = 0$ in the sense that (2.5), (2.6), (2.7) are satisfied.

Then the locally optimal test statistic defined in (1.2), (1.3) is

$$Z^{(t)} = \sum_{1}^{\infty} Z_q^{(t)} \tag{2.15}$$

where

$$Z_q^{(t)} = \frac{1}{q!\mu^q} \int_{[0,t]^q} \chi(t_1, \cdots, t_q) k'(t_1, \cdots, t_q; \mu) \prod_{j=1}^{q} [N(dt_j) - \mu dt_j]. \tag{2.16}$$

The proofs of this and the following theorems are given in the appendix.

In order to apply Corollary 1.4 we require the terms Z_q to be uncorrelated and further for the concept of local efficiency to be reasonable we require these terms to be approximately normally distributed. These results follow as corollaries to the following theorem. Because we may be interested in the performance of terms of the form (2.16) but with functions other than $k'(t_1, \cdots, t_q; \mu)$ we first state a general result.

Theorem 2.2. Suppose that the point process is Poisson with rate μ. Suppose further that $f(t_1,\cdots,t_q)$ is a sequence of functions $(q=1,2,\cdots)$ such that $f(t_1,\cdots,t_q)$ is invariant under permutations of its arguments and

$$\int_{[0,t]^q} [f(t_1,\cdots,t_q)]^2 dt_1,\cdots,dt_q < \infty. \tag{2.17}$$

Let

$$X_q^{(t)} = \frac{1}{q!\,\mu^q}\int_{[0,t]^q} \chi(t_1,\cdots,t_q) f(t_1,\cdots,t_q)\prod_1^q [N(dt_j)-\mu dt_j]. \tag{2.18}$$

Then

$$\begin{aligned} &\text{(i)}\quad E(X_q^{(t)})=0;\\ &\text{(ii)}\quad E(X_q^{(t)}X_r^{(t)})=0 \text{ if } q\neq r;\\ &\text{(iii)}\quad E([X_q^{(t)}]^2)=\frac{1}{q!\,\mu^q}\int_{[0,t]^q}[f(t_1,\cdots,t_q)]^2 dt_1\cdots dt_q. \end{aligned} \tag{2.19}$$

Suppose in addition that $f(t_1+\tau,\cdots,t_q+\tau)=f(t_1,\cdots,t_q)$ for each $q, t_1,\cdots,t_q,\tau$ and

$$\int_{\mathfrak{R}^{q-1}} [f(0,t_2,\cdots,t_q)]^2 dt_2\cdots dt_q < \infty. \tag{2.20}$$

Then

$$\text{(iv)}\quad \lim_{t\to\infty}\frac{1}{t}E([X_q^{(t)}]^2)=\frac{1}{q!\,\mu^q}\int_{\mathfrak{R}^{q-1}}[f(0,t_2,\cdots,t_q)]^2 dt_2\cdots dt_q. \tag{2.21}$$

(v) The random variables $t^{-\frac{1}{2}}X_q^{(t)}$; $q=1,2,\cdots$ are asymptotically independently normally distributed with zero means, and variances given by (2.21).

Corollary 2.3. Suppose that (2.9), (2.11) and (2.12) are satisfied and $Z_q^{(t)}$ is as defined in (2.16). Then, when $\xi=0$, the random variables $t^{-\frac{1}{2}}Z_q^{(t)}$; $q=1,2,\cdots$ are asymptotically independently normally distributed with zero means and variances given by τ_q^2 defined by (2.12).

Corollary 2.4. Suppose that the hypotheses of Theorem 2.1 are satisfied and that the sequence of functions $f(t_1,\cdots,t_q)$: $q=1,2,\cdots$ is defined as in Theorem 2.2, satisfies (2.17) and is normalized so that for each q

$$\begin{aligned}[\sigma_q^{(t)}]^2 &= \frac{1}{q!\,\mu^q}\int_{[0,t]^q} f(t_1,\cdots,t_q)^2 dt_1\cdots dt_q\\ &= \frac{1}{q!\,\mu^q}\int_{[0,t]^q} f(t_1,\cdots,t_q)k'(t_1,\cdots,t_q;\mu)dt_1,\cdots,dt_q.\end{aligned} \tag{2.22}$$

Then the test statistic, with $X_q^{(t)}$ as in (2.18),

$$\sum_{q=1}^{\infty} X_q^{(t)} \tag{2.23}$$

has local efficiency $\Sigma[\sigma_q^{(t)}]^2/\Sigma[\tau_q^{(t)}]^2$ where $[\sigma_q^{(t)}]^2$ is given by (2.22) and $[\tau_q^{(t)}]^2$ by (2.7). In particular

$$S_Q^{(t)} = \sum_{q\in Q} Z_q^{(t)} \tag{2.24}$$

has local efficiency $\Sigma_{q\in Q}[\tau_q^{(t)}]^2/\Sigma_1^{\infty}[\tau_q^{(t)}]^2$. Further, if Q is given then (2.24) maximises the local efficiency amongst statistics of the form (2.23) when the sum is limited to terms with $q \in Q$.

In practice it will be feasible to calculate (2.24) only if it consists of a few terms. Thus it is essential for the success of the tests proposed in this paper that the sum $\Sigma[\tau_q^{(t)}]^2$ be dominated by a small number of terms.

A similar result holds for asymptotic local efficiency: we state the result only for statistics of the form (2.24).

Corollary 2.5. Suppose (2.9), (2.10), (2.11), (2.12) are satisfied. Then the test statistic $t^{-\frac{1}{2}}S_Q^{(t)}$ where $S_Q^{(t)}$ is as in (2.24) and τ_q is given by (2.12) has asymptotic local efficiency

$$\sum_{q\in Q} \tau_q^2 \Big/ \sum_1^{\infty} \tau_q^2. \tag{2.25}$$

Further, under the hypothesis that the process is Poisson with rate μ it is asymptotically normally distributed with zero mean and variance $\Sigma_{q\in Q}\tau_q^2$.

We now consider the more usual situation where μ is unknown. We follow the approach outlined in the introduction. Condition (1.9) is satisfied with $U = n$, the number of *points* observed in time t. Under the assumption of stationarity (2.9) the first term in the series (2.15) can be expressed $k'(0)(n - \mu t)/\mu$. The other terms are uncorrelated with n. So the random variable which maximises the local efficiency subject to (1.10) is

$$\sum_{q=2}^{\infty} Z_q^{(t)} \tag{2.26}$$

where $Z_q^{(t)}$ is given by (2.16). Similarly the local efficiencies of random variables of the form (2.23) or (2.24) with respect to (2.26) are given by

$$\sum [\sigma_q^{(t)}]^2 \Big/ \sum_{q=2}^{\infty} [\tau_q^{(t)}]^2$$

$$\sum_{q\in Q}[\tau_q^{(t)}]^2 \Big/ \sum_{q=2}^{\infty}[\tau_q^{(t)}]^2$$

with corresponding formulae for asymptotic local efficiencies. However even with the $q = 1$ term deleted variables (2.23) and (2.24) still depend on μ. The next theorem shows that these expressions are asymptotically equivalent to those obtained when μ is replaced by $\hat{\mu} = n/t$ provided (2.23) and (2.24) consist only of a finite number of terms and do not include a $q = 1$ term.

Theorem 2.6. Suppose $f(t_1,\cdots,t_q;\mu)$ is invariant under permutation of its first q arguments and

$$f(t_1+\tau,\cdots,t_q+\tau;\mu)=f(t_1,\cdots,t_q;\mu)$$

for each $t_1,\cdots,t_q,\tau$ and $\mu>0$. Suppose

(2.27) $$\lim_{\mu\to\mu_0}\int_{\mathfrak{R}^{q-1}}[f(0,t_2,\cdots,t_q;\mu)-f(0,t_2,\cdots,t_q;\mu_0)]^2dt_2\cdots dt_q=0$$

and

$$\int_{\mathfrak{R}^{q-1}}[f(0,t_2,\cdots,t_q;\mu_0)]^2dt_2\cdots dt_q<\infty$$

for some $\mu_0>0$. Let $\hat{\mu}=n/t$. Then, provided $q>1$, under the hypothesis that the process is Poisson with rate μ_0,

(2.28) $$t^{-\frac{1}{2}}\left|\int_{[0,t]^q}f(t_1,\cdots,t_q;\hat{\mu})\prod_1^q[N(dt_j)-\hat{\mu}dt_j]\right.$$
$$\left.-\int_{[0,t]^q}f(t_1,\cdots,t_q;\mu_0)\prod_1^q[N(dt_j)-\mu_0dt_j]\right|$$

$\to 0$ in probability as $t\to\infty$.

Following the discussion in the introduction we have the following result.

Corollary 2.7. Suppose that the hypotheses of Corollary 2.5 and also (2.13) are satisfied for each $\mu>0$. Then the statistic

(2.29) $$t^{-\frac{1}{2}}\sum_{q\in Q}\frac{1}{q!\hat{\mu}^q}\int_{[0,t]^q}\chi(t_1,\cdots,t_q)k'(t_1,\cdots,t_q;\hat{\mu})\prod_1^q[N(dt_j)-\hat{\mu}dt_j]$$

where Q is a finite subset of $\{2,3,\cdots\}$ has asymptotic local efficiency

(2.30) $$\sum_{q\in Q}\tau_q^2\Big/\sum_{q=2}^{\infty}\tau_q^2$$

when compared with random variables (depending on μ) which are uncorrelated with n.

Further, under the Poisson hypothesis (2.29) is asymptotically normally distributed with zero mean and variance $\Sigma_{q\in Q}\tau_q^2$.

3. Inference via the spectrum

We now relate terms of the form (2.16) or (2.18) with terms related to the cumulant spectra and periodograms of various orders. Cumulant spectra and periodograms can be defined for continuous and discrete processes (see, for example, Brillinger (1975)) and these terms may be extended naturally to point processes (see Brillinger (1972)). It is convenient, however, to adopt slightly different definitions. We consider an analogue of the qth-order spectrum

(3.1) $$\kappa(\lambda_1,\cdots,\lambda_q)=\int_{\mathfrak{R}^{q-1}}k(t_1,t_2,\cdots,t_q)\exp\left(2\pi i\sum_{j=1}^{q}\lambda_jt_j\right)dt_2\cdots dt_q$$

when $\lambda_1+\cdots+\lambda_q=0$ and an analogue of the qth-order periodogram

(3.2) $$I_t(\lambda_1,\cdots,\lambda_q)=\int_{[0,t]^q}\chi(t_1,\cdots,t_q)\exp\left(2\pi i\sum_{j=1}^{q}\lambda_jt_j\right)\prod_{1}^{s}[N(dt_j)-\mu dt_j].$$

Under the stationarity assumption (3.1) does not depend on t_1 when $\Sigma\lambda_i=0$. Note that $I_t(\lambda_1,\cdots,\lambda_q)$ does not depend on μ if none of the λ_i are zero and is equal to zero if μ is replaced by $\hat{\mu}$ and any of the λ_i are zero.

The relation we require is given by the following theorem, which is proved in the appendix.

Theorem 3.1. Suppose $f(t_1,\cdots,t_q)$ is invariant under permutation and translation of its arguments and

(3.3) $$\int_{\mathfrak{R}^{q-1}}|f(0,t_2,\cdots,t_q)|^r dt_2\cdots dt_q<\infty$$

for $r=1,2$. Let

(3.4) $$\phi(\lambda_1,\cdots,\lambda_q)=\int_{\mathfrak{R}^{q-1}}f(t_1,\cdots,t_q)\exp\left(2\pi i\sum_{j=1}^{q}\lambda_jt_j\right)dt_2\cdots dt_q$$

when $\lambda_1+\cdots+\lambda_q=0$, and suppose

(3.5) $$\sup_{\lambda_1+\cdots+\lambda_q=0}\prod_{1}^{q}(1+|\lambda_i|)|\phi(\lambda_1,\cdots,\lambda_q)|^2<\infty.$$

Then, if N is generated by a Poisson process with rate μ,

$$\frac{1}{t}E\left|\int_{[0,t]^q}\chi(t_1,\cdots,t_q)f(t_1,\cdots,t_q)\prod_1^q[N(dt_j)-\mu dt_j]\right.$$

$$(3.6)\qquad \left. -t^{-(q-1)}\sum_{|l_1+\cdots+l_q=0}\cdots\sum\phi(l_1/t,\cdots,l_q/t)\bar{I}_t(l_1/t,\cdots,l_q/t)\right|^2$$

$$\to 0 \quad \text{as} \quad t\to\infty.$$

Analogues of Corollaries 2.5 and 2.7 can now be given—we state only the analogue of 2.7.

Corollary 3.2. Suppose (2.9), (2.10), (2.11), (2.12), (2.13) are satisfied, Q is a finite subset of $\{2,3,\cdots\}$ and

$$(3.7)\qquad \sup_{\mu_1<\mu<\mu_2}\int_{\mathfrak{R}^{q-1}}|k'(0,t_2,\cdots,t_q;\mu)|\,dt_2\cdots dt_q<\infty$$

for each $q\in Q$ and $\mu_2>\mu_1>0$. Let

$$(3.8)\qquad \kappa'(\lambda_1,\cdots,\lambda_q;\mu)=\int_{\mathfrak{R}^{q-1}}k'(t_1,\cdots,t_q;\mu)\exp\left(2\pi i\sum_{j=1}^{\infty}\lambda_j t_j\right)dt_2\cdots dt_q$$

and suppose for each $q\in Q,\mu_2>\mu_1>0$ that

$$(3.9)\qquad \sup_{\Sigma\lambda_i=0}\ \sup_{\mu_1<\mu<\mu_2}\prod_1^q(1+|\lambda_i|)|\kappa'(\lambda_1,\cdots,\lambda_q;u)|<\infty.$$

Then the statistic

$$(3.10)\qquad t^{-\frac{1}{2}}\sum_{q\in Q}\frac{t^{-(q-1)}}{q!\hat{\mu}^q}\sum_{|l_1+\cdots+l_q=0|}\cdots\sum\kappa'(l_1/t,\cdots,l_q/t;\hat{\mu})\bar{I}_t(l_1/t,\cdots,l_q/t;\hat{\mu}),$$

where $I_t(l_1/t,\cdots,l_q/t;\hat{\mu})$ is given by (3.2) with μ replaced by $\hat{\mu}$, has asymptotic local efficiency

$$(3.11)\qquad \sum_{q\in Q}\tau_q^2\Big/\sum_{q=2}^{\infty}\tau_q^2$$

when compared with random variables (depending on μ) which are uncorrelated with n. τ_q^2 is given by (2.12) and is equal to

$$(3.12)\qquad \frac{1}{q!\mu^q}\int_{\mathfrak{R}^{q-1}}\left|\kappa'\left(-\sum_2^q\lambda_j,\lambda_2,\cdots,\lambda_q;\mu\right)\right|^2 d\lambda_2\cdots d\lambda_q.$$

Further, under the Poisson hypothesis (3.10) is asymptotically normally distributed with zero mean and variance $\Sigma_{q\in Q}\tau_q^2$.

The uniform bounds (3.7) and (3.9) are to ensure that μ can be replaced by $\hat{\mu}$. The efficiencies of tests of the form (3.3) but with κ' replaced by some other function ϕ can be found with the aid of Corollary 1.4. Details are left to the reader.

The statistic (3.10) is not suitable for calculation unless $\kappa'(\lambda_1, \cdots, \lambda_q)$ decays to zero very quickly. However an approximation may be found which may be evaluated using the fast Fourier transform computer program (see Brigham (1974)). We illustrate with the $q = 2$ term.

Suppose that the interval $(0, t)$ is divided into r intervals of length Δ $(t = r\Delta)$. To enable efficient use of the fast Fourier transform r will normally be a power of 2. Let $N_j = N[j\Delta, (j+1)\Delta]$, the number of points falling into the interval $[j\Delta, (j+1)\Delta]$; $j = 0, 1, \cdots, r-1$. Suppose that $f_j : j = 0, 1, \cdots,$ is an infinite sequence and $f(s_1, s_2)$ is a function defined by

$$f(s_1, s_2) = f_{|j_1 - j_2|} \quad \text{if} \quad s_i \in (j_i\Delta, (j_i + 1)\Delta). \tag{3.13}$$

Then the expression

$$\begin{aligned} &\sum_{i=0}^{r-1}\sum_{j=0}^{r-1} f_{|i-j|}(N_i - \mu\Delta)(N_j - \mu\Delta) - nf_0 \\ &= \int_0^t \int_0^t \chi(s_1, s_2) f(s_1, s_2)[N(ds_1) - \mu ds_1][N(ds_2) - \mu ds_2] \end{aligned} \tag{3.14}$$

is of the form (2.18) with $q = 2$.

The local efficiency of (3.14) is optimized by choosing

$$f_j = k_j' = \int_{-\Delta}^{\Delta} (\Delta - |s|)k'(0, j\Delta + s)ds / \Delta^2, \tag{3.15}$$

the local efficiency with respect to the best $q = 2$ term being

$$\begin{aligned} &1 - \sum_{j=-r}^{r} (r - |j|) \int_{-\Delta}^{\Delta} (\Delta - |s|)[k'(0, s + j\Delta) - k_j']^2 ds \Big/ \\ &\int_{-t}^{t} (t - |s|)[k'(0, s)]^2 ds. \end{aligned} \tag{3.16}$$

Thus one will choose r sufficiently large to make (3.16) close to 1. The function f defined by (3.13) is not invariant under translation of its arguments. However this condition is not required for the small sample results and analogues of the asymptotic results can be obtained.

Let

$$d_l = r^{-\frac{1}{2}} \sum_{j=0}^{r-1} (N_j - \mu\Delta)\exp(2\pi ijl/r) \tag{3.17}$$

and

$$\mathcal{K}(\lambda) = 2\sum_{j=1}^{\infty} k'_j \cos(2\pi i j\lambda). \tag{3.18}$$

Following Theorem 4.8 of Davies (1973) one can show that as $t \to \infty$, with Δ fixed and with f_j replaced by k'_j, (3.14) is asymptotically equivalent to

$$\sum_{j=0}^{r-1} [(N_j - \mu\Delta)^2 - N_j]k'_0 + \sum_{l=0}^{r-1} |d_l|^2 \mathcal{K}(l/r). \tag{3.19}$$

This is the suggested version of the $q = 2$ term.

4. Examples

(i) *Gauss–Poisson process.* We observe a point process consisting of the superposition of two processes. The first is a Poisson process with rate μ. The second process is generated as follows: each *point* in the first process, with probability ξ, independently gives rise to a *point* in the second process, the distance between each initiating *point* and the corresponding generated *point* being distributed with a known probability density function $\gamma(\cdot)$. When $\xi = 0$ the process is Poisson with rate μ and when $\xi > 0$ it is of the so-called Gauss–Poisson type (see, for example, Milne and Westcott (1972)). We wish to test the hypothesis $\xi = 0$ against the alternative $\xi > 0$. The probability generating functional may be derived:

$$\log E\prod [u(T_i)] = \mu(1+\xi)\int [u(s)-1]ds$$
$$+ \mu\xi \int\int [u(s_1)-1][u(s_2)-1]\gamma(s_1 - s_2)ds_1 ds_2.$$

One may then calculate the right-hand side of (2.3) and hence find

$$k(s) = \mu(1+\xi), \qquad k(s_1, s_2) = \mu\xi[\gamma(s_1 - s_2) + \gamma(s_2 - s_1)], \qquad k(s_1, \cdots, s_q) = 0$$

if $q > 2$. Hence, in this case, only the first two terms of the series (2.15) are non-zero and if μ is unknown one will consider

$$\frac{t^{-\frac{1}{2}}}{2\hat{\mu}} \int_{s_1=0}^{t} \int_{s_2=0}^{t} \chi(s_1, s_2)[\gamma(s_1 - s_2) + \gamma(s_2 - s_1)] \prod_{1}^{2} [N(ds_i) - \hat{\mu} ds_i] \tag{4.1}$$

which will have asymptotic local efficiency equal to 1 when compared with variables uncorrelated with n. Further under the Poisson hypothesis (4.1) is asymptotically normally distributed with zero mean and variance

$$\frac{1}{2\mu} \int_{s=-\infty}^{\infty} [\gamma(s) + \gamma(-s)]^2 ds.$$

Expression (4.1) is asymptotically equivalent to

$$\frac{t^{-\frac{1}{2}}}{2\hat{\mu}} \sum_{l \neq 0} \kappa'(l/t, -l/t) I_t(l/t, -l/t)$$

where I_t is given by (3.2) and

$$\kappa'(\lambda, -\lambda) = \int_{s=-\infty}^{\infty} [\gamma(s) + \gamma(-s)] e^{2\pi i \lambda s} ds.$$

A test commonly used for detecting deviations from the Poisson distribution is obtained by dividing the interval of observation, $(0, t)$, into subintervals of length Δ say and then calculating the ratio of the variance to the mean of the numbers of *points* occurring in the sub-intervals. It is of interest to find the efficiency of this test for the example. We use the notation of the last part of Section 3. The appropriate random variable to be considered is

$$\sum_{i=0}^{r-1} (N_i - \mu\Delta)^2 - n. \tag{4.2}$$

This is just (3.17) with $f_0 = 1$ and $f_i = 0$ if $i \neq 0$ and hence the asymptotic local efficiency can be calculated:

$$\Delta^{-3} \left\{ \int_{-\Delta}^{\Delta} (\Delta - |s|)[\gamma(s) + \gamma(-s)] ds \right\}^2 \bigg/ \int_{-\infty}^{\infty} [\gamma(s) + \gamma(-s)]^2 ds. \tag{4.3}$$

The efficiency is highest when $\gamma(s) + \gamma(-s) = 2(\Delta - |s|)/\Delta^2$ if $|s| < \Delta$, 0 otherwise, and then it is $\frac{2}{3}$.

In this and the following examples γ might be known up to an unknown scale factor. For example in the 'triangular' form for γ just considered Δ might be unknown. The present theory may be extended to this situation in the manner suggested by Davies (1977).

(ii) *Self-exciting (or epidemic) process.* We consider a point process which at any time t_0 has conditional rate, given the past,

$$\mu + \xi \int_{-\infty}^{t_0} \gamma(t_0 - s) N(ds)$$

where $\gamma(s)$ is a probability density function. This process may be considered as a simple model for the infection times in a population where there is contagion and also a spontaneous generation of infection. In effect one has a primary Poisson process of rate μ, each *point* of which gives rise to a branching process of secondary points. This process has been discussed by Hawkes (1971) and optimal tests of the hypothesis $\xi = 0$ (no contagion) against $\xi = O(t^{-\frac{1}{2}})$ have been considered by Yang (1968).

The locally optimal test considers arbitrarily small ξ and so the secondary processes die out very quickly. In fact in time t only $O(\xi^2 t)$ 'epidemics' with more than one secondary case will occur and, when compared with the $O(\xi t)$ 'epidemics' with one secondary case, can be ignored. Hence as far as the locally optimal test is concerned the problem reduces to that in (i).

(iii) *Neyman–Scott cluster process.* We suppose there is an unobserved primary Poisson process with rate ξ and that given this process the observed process is Poisson with rate at time t_0

$$\mu+\alpha\int_{-\infty}^{\infty}\gamma(t_0-s)M(ds) \tag{4.4}$$

where M is the measure corresponding to the primary process and γ is a (known) probability density function. This type of process has been considered in a wide variety of situations, see for example, Vere-Jones (1970). The present example is a super-position of a Poisson process with rate μ and a cluster process; the clusters having an average of α *points* each and occurring at rate ξ. If we wish to test the hypothesis $\alpha=0$ against $\alpha>0$ the problem again reduces to the Gauss–Poisson situation. However if α is supposed to be a known parameter and we wish to test $\xi=0$ against $\xi>0$ clusters with more than two elements will be significant. The probability generating functional is given by

$$\log E\prod[u(T_i)]=\mu\int[u(s)-1]ds$$
$$+\xi\int\left\{\exp\left[\alpha\int(u(s+x)-1)\gamma(x)dx\right]-1\right\}ds$$

and hence $k(s)=\mu+\xi\alpha$ and for $q>1$

$$k(s_1,\cdots,s_q)=\xi\alpha^q\int_{-\infty}^{\infty}\prod_1^q\gamma(s_i-x)dx \tag{4.5}$$

and so all terms in (2.15) contain information. One can also calculate (for $q>1$)

$$\kappa(\lambda_1,\cdots,\lambda_q)=\xi\alpha^q\prod_{j=1}^{q}\int_{s=-\infty}^{\infty}\gamma(s)\exp(2\pi i\lambda_j s)ds.$$

Thus, in principle, expressions (2.29) and (3.13) could be evaluated. In practice one would be unwilling to evaluate more than the $q=2$ term although it would be possible to use the third- and fourth-order terms particularly if an approximation similar to (3.22) was used. In order to find the efficiency of the resulting test it is necessary to find τ_q^2 defined in (2.12). In the present example, for $q>1$,

$$\tau_q^2=\frac{\alpha^{2q}}{q!\,\mu^q}\int_{x=-\infty}^{\infty}\left[\int_{s=-\infty}^{\infty}\gamma(s-x)\gamma(s)ds\right]^q dx. \tag{4.6}$$

For definiteness we will consider a particular value of γ: $\gamma(s)=1/\Delta$ if $0<s<\Delta$ and 0 otherwise. Then

(4.7) $$\tau_q^2 = 2\alpha^{2q} / [(q+1)!\,\mu^q \Delta^{q-1}].$$

Thus if $\alpha^2/(\mu\Delta)$ is less than one most of the information will be in the second-order term. This corresponds to a small cluster size or a cluster length long compared with $1/\mu$. On the other hand if $\alpha^2/(\mu\Delta)$ is large compared with one, tests based on low-order terms will be very inefficient. In marginal cases it may be worth calculating third- and fourth-order terms. However α will have to be known in order to obtain the correct weighting of these terms.

Acknowledgements

I wish to thank Professors Brillinger and Vere-Jones and the referee for their suggestions and encouragement.

Appendix: Proofs of the theorems

Proof of Theorem 2.1. To simplify the notation we suppress the μ in the argument lists of the various functions considered. Since $k(t_1,\cdots,t_q;\xi)$ is invariant under permutation of its arguments, $t_1,\cdots,t_q$, it is convenient to consider k as a set function defined for finite sets:

$$k(S;\xi) = k(t_1,\cdots,t_q;\xi)$$

if $S = \{t_1,\cdots,t_q\}$.

Suppose that for some $\delta > 0$

(A1) $$\sum_{q=0}^{\infty} \frac{(1+\delta)^q}{q!} \int_{[0,t]^q} |k(t_1,\cdots,t_q;\xi)|\,dt_1\cdots dt_q < \infty.$$

Then it can be shown that the probability density for the *points* occurring in $(0,t)$: $T_1,\cdots,T_n$, with respect to Lebesgue measure on $\Sigma\mathcal{R}^q$ is given by

(A2) $$p(T_1,\cdots,T_n;\xi) = e^{h(\xi)} \sum \prod_{j=1}^{r} h(S_j;\xi)$$

where the sum is over all partitions $\{S_1,\cdots,S_r\}$ of the set $\{T_1,\cdots,T_n\}$ and

(A3) $$h(\xi) = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \int_{[0,t]^j} k(u_1,\cdots,u_j;\xi)\,du_1\cdots du_j,$$

(A4) $$h(t_1,\cdots,t_q;\xi) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_{[0,t]^j} k(t_1,\cdots,t_q,u_1,\cdots,u_j;\xi)\,du_1\cdots du_j.$$

This result was given by Kuznetsov and Stratonovich (1956). A rigorous proof can be given along the lines of Macchi (1975). Our first lemma, which we state without proof, concerns the differentiability of the function h just defined.

Lemma A1. Suppose (2.5), (2.6) and (2.7) are satisfied and $h(t_1,\cdots,t_q;\xi)$ is defined by (A3) if $q=0$ and (A4) otherwise. Then

(A5)
$$\lim_{\xi\to 0}\sum_{q=0}^{\infty}\frac{1}{q!}\int_{[0,t]^q}|[h(t_1,\cdots,t_q;\xi)-h(t_1,\cdots,t_q;0)]/\xi - h'(t_1,\cdots,t_q)|\,dt_1\cdots dt_q=0$$

where

(A6)
$$h'(t_1,\cdots,t_q)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\int_{[0,t]^j}k'(t_1,\cdots,t_q,u_1,\cdots,u_j)du_1\cdots du_j$$

(the initial term being omitted when $q=0$). Further

(A7)
$$\sum_{q=0}^{\infty}\frac{1}{q!}\int_{[0,t]^q}|h'(t_1,\cdots,t_q)|\,dt_1\cdots dt_q<\infty.$$

Another lemma, which we will state without proof, follows.

*Lemma A*2. Suppose $\{c_j : j=1,2,\cdots\}$ is an infinite sequence satisfying

$$\sum|c_j|/j!<\infty.$$

Then

$$\exp\left[\sum_{j=1}^{\infty}c_j/j!\right]=\sum_{q=0}^{\infty}\frac{1}{q!}\sum_{l=1}^{q}\sum_{(i)}\prod_{1}^{l}c_{|S_j|}$$

where the sum (i) is over all partitions $\{S_1,\cdots,S_l\}$ of $\{1,\cdots,q\}$ and $|S|$ indicates the number of elements in the set S.

The next lemma concerns the differentiability of p.

Lemma A3. Suppose (2.4), (2.5), (2.6), (2.7) are satisfied, $p(t_1,\cdots,t_q;\xi)$ is as defined by (A2) and

(A8)
$$p'(t_1,\cdots,t_q)=e^{-\mu t}\sum_{j=0}^{\infty}\sum_{(i)}\mu^{q-j}h'(S_j)$$

where the sum (i) is over all subsets (including the empty subset) S_j of $\{t_1,\cdots,t_q\}$ with j elements. Then

(A9)
$$\lim_{\xi\to 0}\sum_{q=0}^{\infty}\int_{0<t_1<\cdots<t_q<t}\cdots\int|[p(t_1,\cdots,t_q;\xi)-p(t_1,\cdots,t_q;0)]/\xi - p'(t_1,\cdots,t_q)|\,dt_1\cdots dt_q=0.$$

Proof.

$$p(t_1,\cdots,t_q;\xi)-p(t_1,\cdots,t_q;0)$$
$$=\exp(\theta h(\xi)+(1-\theta)h(0))\sum\prod_{j=1}^{r}[\theta h(S_j;\xi)+(1-\theta)h(S_j;0)]\ -\mu^n e^{-\mu t}$$

where the sum is over all partitions $\{S_1, \cdots, S_r\}$ of $\{t_1, \cdots, t_q\}$ and $\theta = 1$. Expand in a Taylor series in θ to obtain

$$e^{-\mu t} \sum_{(i)} \mu^{q-j} [h(t_{\nu_1}, \cdots, t_{\nu_j}; \xi) - h(t_{\nu_1}, \cdots, t_{\nu_j}; 0)]$$

$$+ \frac{1}{2} \exp(\theta h(\xi) + (1-\theta) h(0)) \sum_{(ii)} [h(R_1; \xi) - h(R_1; 0)] [h(R_2; \xi) - h(R_2; 0)]$$

$$\cdot \prod_{j=1}^{r} [\theta h(S_j; \xi) - (1-\theta) h(S_j; 0)]$$

where the sum (i) is over all subsets $(\nu_1, \cdots, \nu_j)$ of $\{1, \cdots, q\}$, the sum (ii) is over all partitions $\{R_1, R_2, S_1, \cdots, S_r\}$ of $\{t_1, \cdots, t_q\}$, and $0 < \theta < 1$.

Thus (A9) is bounded by

$$e^{-\mu t} \sum_{q=0}^{\infty} \frac{1}{q!} \sum_{j=0}^{q} \binom{q}{j} (\mu t)^{q-j} \int_{[0,t]^j} |[h(t_1, \cdots, t_j; \xi) - h(t_1, \cdots, t_j; 0)]/\xi$$

$$- h'(t_1, \cdots, t_j)| \, dt_1 \cdots dt_j$$

$$+ \frac{1}{2\xi} \exp(\theta h(\xi) + (1-\theta) h(0)) \cdot \sum_{q=0}^{\infty} \frac{1}{q!}$$

$$\cdot \sum_{i=0}^{q} \binom{q}{j} \int_{[0,t]^i} |h(t_1, \cdots, t_i; \xi) - h(t_1, \cdots, t_i; 0)| \, dt_1, \cdots, dt_i$$

$$\cdot \sum_{j=0}^{q-i} \binom{q-i}{j} \int_{[0,t]^j} |h(t_1, \cdots, t_j; \xi) - h(t_1, \cdots, t_j; 0)| \, dt_1 \cdots dt_j$$

$$\cdot \sum_{(iii)} \int_{[0,t]^{q-i-j}} \prod_{l=1}^{q-i-j} |\theta h(S_l; \xi) - (1-\theta) h(S_l; 0)| \, dt_1 \cdots dt_{q-i-j}$$

where the sum (iii) is over all partitions $\{S_1, \cdots, S_l\}$ of $\{t_1, \cdots, t_{q-i-j}\}$. Further simplification involving changing the order of summation and applying Lemma A2 leads to the bound

$$\sum_{j=0}^{\infty} \frac{1}{j!} \int_{[0,t]^j} |[h(t_1, \cdots, t_j; \xi) - h(t_1, \cdots, t_j; 0)]/\xi - h'(t_1, \cdots, t_j)| \, dt_1 \cdots dt_j$$

$$+ \frac{\xi}{2} \exp(\theta h(\xi) + (1-\theta) h(0)) \cdot \Big[\frac{1}{\xi} \sum_{j=0}^{\infty} \frac{1}{j!} \int_{[0,t]^j} |h(t_1, \cdots, t_j; \xi)$$

$$- h(t_1, \cdots, t_j; 0)| \, dt_1 \cdots dt_j \Big]^2$$

$$\cdot \exp\Big[\sum_{j=1}^{\infty} \frac{1}{j!} \int_{[0,t]^j} \Big| \theta h(t_1, \cdots, t_j; \xi) - (1-\theta) h(t_1, \cdots, t_j; 0) \Big| \, dt_1 \cdots dt_j \Big]$$

and this tends to zero as $\xi \to 0$. This completes the proof of the lemma.

We are now in a position to prove Theorem 2.1. Condition (2.6) implies that (A1) is satisfied. Now $p(t_1,\cdots,t_q;0)=\mu^q e^{-\mu t}$ so that, in view of (A9), the locally optimal test statistic is

(A10) $$Z^{(t)}=\sum_{j=0}^{\infty}\sum_{(i)}\mu^{-j}h'(S_j)$$

where the sum (i) is over all subsets of j elements of $\{T_1,\cdots,T_n\}$. $Z^{(t)}$ may be expressed as

$$Z^{(t)}=\sum_{j=0}^{\infty}\frac{1}{j!\,\mu^j}\int_{[0,t]^j}\chi(t_1,\cdots,t_j)h'(t_1,\cdots,t_j)\prod_1^j N(dt_i)$$

where N is as defined in Section 2. Expanding $h'(t_1,\cdots,t_j)$ according to (A3) and (A4) and changing the order of summation we find

$$Z^{(t)}=\sum_{q=1}^{\infty}\frac{1}{\mu^q}\int_{[0,t]^q}\chi(t_1,\cdots,t_q)k'(t_1,\cdots,t_q)\sum_{j=0}^{q}\frac{(-1)^j}{j!(q-j)!}\prod_1^{q-j}N(dt_i)\prod_{q-j+1}^{q}\mu dt_i$$

(A11) $$=\sum_{q=1}^{\infty}\frac{1}{q!\,\mu^q}\int_{[0,t]^q}\chi(t_1,\cdots,t_q)k'(t_1,\cdots,t_q)\prod_{j=1}^{q}[N(dt_j)-\mu dt_j]$$

and this completes the proof of Theorem 2.1.

Proof of Theorem 2.2. Statements (i), (ii) and (iii) may be proved rigorously by expanding the differential parts of (2.18) to form an expression of the form (A10) and taking expectations conditionally on n being given. Alternatively heuristic proofs may be given by taking expectations of the differential parts. Statement (iv) is proved by application of the Lebesgue convergence theorem. To prove (v) one applies the central limit theorem to the sum of integrals of the form (2.18) but taken over $[\nu\tau,(\nu+1)\tau]^q:\nu=0,1,\cdots,[t/\tau]$ for fixed τ with t tending to infinity and then shows that for large τ this sum approximates (2.18).

We now proceed with a sketch of the proof of Theorem 2.6, but first we need two lemmas which we state without proof.

Lemma A4. Suppose

(A12) $$x_{l,q,r}=\sum\prod_1^l\nu_i$$

where the sum is over all subsets $\{\nu_1\cdots\nu_l\}$ of l elements of $\{q,\cdots,r\}$. Then $x_{l,q,r}$ is a polynomial of degree $2l$ in r.

Lemma A5. Suppose $f(t_1,\cdots,t_q)$ is invariant under permutation or translation of its arguments and satisfies (2.20). Then

(A13) $$t^{-(q-j+1)}\int_{[0,t]^j}\left[\int_{[0,t]^{q-j}}f(t_1,\cdots,t_q)\prod_{j+1}^{q}dt_i\right]^2\prod_1^j dt_i$$

is bounded by the integral (2.20). Further if $q > j > 0$ or $q > 1, j = 0$ then (A13) tends to zero as t tends to infinity.

Expression (2.28) is bounded by the sum of the two terms:

$$\text{(A14)}\quad t^{-\frac{1}{2}}\left|\int_{[0,t]^q} [f(t_1,\cdots,t_q;\hat{\mu})-f(t_1,\cdots,t_q;\mu)]\prod_1^q[N(dt_i)-\hat{\mu}dt_i]\right|$$

and

$$\text{(A15)}\quad t^{-\frac{1}{2}}\left|\int_{[0,t]^q} f(t_1,\cdots,t_q;\mu)\left\{\prod_1^q[N(dt_i)-\hat{\mu}dt_i]-\prod_1^q[N(dt_i)-\mu dt_i]\right\}\right|.$$

The second term (A15) can be expressed

$$\sum_{j=0}^{q-1}[t^{\frac{1}{2}}(\mu-\hat{\mu})]^{q-j}\binom{q}{j}t^{-\frac{1}{2}(q-j+1)}\int_{[0,t]^q} f(t_1,\cdots,t_q;\mu)\prod_1^j[N(dt_i)-\mu dt_i]\prod_{j+1}^q dt_i.$$

Taking the variance of the jth integral, it follows from (A13) that (A15) tends to zero in probability as $t\to\infty$ provided $q>1$.

We consider the conditional expectation of the square of (A14) given n. After expanding the differential part and considering the n *points* as being independently uniformly distributed on $(0, t)$ and carrying out considerable manipulation one obtains

$$\text{(A16)}\quad \begin{aligned} &\frac{1}{t}\sum_{j=0}^{q}\int_{[0,t]^j}\left[\int_{[0,t]^{q-j}}\{f(t_1,\cdots,t_q;\hat{\mu})\right.\\ &\qquad\qquad\left.-f(t_1,\cdots,t_q;\mu)\}dt_{j+1}\cdots dt_q\right]^2 dt_1\cdots dt_j\\ &\quad [q!/(q-j)!]^2\binom{n}{j}n^{-j}\hat{\mu}^{2q-j}\\ &\quad\cdot\sum_{r=0}^{q-j}\sum_{s=0}^{q-j}\binom{q-j}{r}\binom{q-j}{s}(-1)^{r+s}\left[\left(1-\frac{j}{n}\right)\cdots\left(1-\frac{j+r+s-1}{n}\right)\right].\end{aligned}$$

In the final product (2.48) the coefficient of $n^{-\nu}$ is a polynomial of degree 2ν in $(r+s)$. This will cancel when the sum over r and s is formed if $\nu < q-j$. Thus the coefficient of the jth integral is of the order $O(t^{-1}n^{-(q-j)})$ and thus (A16) tends to zero in probability as $t\to\infty$ and $\hat{\mu}\to\mu$. This completes the sketch of the proof of Theorem 2.6.

Proof of Theorem 3.1. Let

$$g_t(t_1,\cdots,t_q)=\sum_{\nu_2=-\infty}^{\infty}\cdots\sum_{\nu_q=-\infty}^{\infty}f(t_1,t_2+\nu_2 t,\cdots,t_q+\nu_q t).$$

Then g_t is integrable on $[0,t]^q$. Further

$$\frac{1}{t}\int_{[0,t]^q} g_t(t_1,\cdots,t_q)\exp\Big(2\pi i\sum_{j=1}^{q} l_j t_j/t\Big)dt_1\cdots dt_q$$
$$=\frac{1}{t}\int_{t_1=0}^{t}\exp\Big(2\pi i\sum_{j=1}^{q} l_j t_1/t\Big)\int_{\mathfrak{R}^{q-1}} f(t_1,\cdots,t_q)$$
$$\exp\Big[2\pi i\Big(-\sum_{j=2}^{q} l_j t_1/t+\sum_{j=2}^{q} l_j t_j/t\Big)\Big]dt_1\cdots dt_q$$

$=0$ if $\Sigma l_j\neq 0$ and $\phi(l_1/t,\cdots,l_q/t)$ otherwise. In view of (3.5)

$$\sum_{l_1+\cdots+l_q=0}\cdots\sum|\phi(l_1/t,\cdots,l_q/t)|^2<\infty$$

and hence by the Fourier inversion theorem

$$g_t(t_1,\cdots,t_q)=t^{-(q-1)}\sum_{l_1+\cdots+l_q=0}\cdots\sum\phi(l_1/t,\cdots,l_q/t)\exp\Big(-2\pi i\sum_{j=1}^{q} l_j t_j/t\Big).$$

By decomposing the differential term one can show

$$\int_{[0,t]^q}\chi(t_1,\cdots,t_q)g_t(t_1,\cdots,t_q)\prod_1^q[N(dt_j)-\mu dt_j]$$
$$=t^{-(q-1)}\sum_{l_1+\cdots+l_q=0}\cdots\sum\phi(l_1/t,\cdots,l_q/t)\bar{I}_t(l_1/t,\cdots,l_q/t).$$

Thus (3.6) is equal to

(A17) $$t^{-1}\mu^q q!\int_{[0,t]^q}[f(t_1,\cdots,t_q)-g_t(t_1,\cdots,t_q)]^2dt_1\cdots dt_q.$$

Now

$$t^{-1}\int_{[0,t]^q}|f(t_1,\cdots,t_q)-g_t(t_1,\cdots,t_q)|\,dt_1\cdots dt_q\to 0 \text{ as } t\to\infty$$

and also

$$t^{-1}\int_{[0,t]^q}[g_t(t_1,\cdots,t_q)]^2dt_1\cdots dt_q$$
$$=t^{-(q-1)}\sum_{l_1+\cdots+l_q=0}\cdots\sum|\phi(l_1/t,\cdots,l_q/t)|^2|(l_1/t,\cdots,l_q/t)|^2$$
$$\to\int_{\mathfrak{R}^{q-1}}|\phi(-(\lambda_2+\cdots+\lambda_q),\lambda_2,\cdots,\lambda_q)|^2d\lambda_2\cdots d\lambda_q$$
$$=\int_{\mathfrak{R}^{q-1}}[f(0,t_2,\cdots,t_q)]^2dt_2\cdots dt_q.$$

These two results together with Theorem 2.2 (iv) imply that (A17) tends to zero as $t \to \infty$. This completes the proof.

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